Micro 2 -Problem Set 2

Question 1:

Consider a firm that produces using a single input - labor, and whose production function is denoted by $f(l)$. Moreover assume that the firm is a price taker, i.e., the firm faces a fixed price for its output, denoted by p, and a fixed price for the labor it uses, w. Define the firm's optimal profit function Π^* to be the profit that it makes when it chooses its quantity to be q^* - the quantity that maximizes its profit. Mathematically $\Pi^*(w,p) = p \cdot q^* - C(q^*)$, where $C(q^*) = w \cdot f^{-1}(q^*)$ and q^* is the quantity that the firm chooses to maximize its profit. Explain why the following properties of the optimal profit function are true:

1. The optimal profit function is homogenous of a degree 1. That is if we look at prices p and w and multiply them by some positive number $t > 0$ then the optimal profit is also multiplied by t.

2. The optimal profit function is increasing in p.

3. The optimal profit function is decreasing in w.

Question 2:

1. Consider the following production functions. For each function derive the isoquant curves $f(l,k) = q_0$ as a function $g(l)$ whose independent variable is l and its dependent variable is k, and present its derivative. Moreover, derive the RTS of the production function.
(a) $f(l,k) = l^\alpha k^\beta$

Question 2:

Consider a firm that produces using a single input - labor, and whose production function is denoted by $f(l)$. Moreover assume that the firm is a price taker, i.e, that face fixed prices for its output, denoted by p, and for the labor it uses, w. Define the firm's optimal profit function Π^* to be the profit that it makes when it chooses its quantity to be q^* - the quantity that maximizes its profit. Mathematically $\Pi^*(w,p) = p \cdot q^* - C(q^*)$, where $c(q^*) = w \cdot f^{-1}(q^*)$ and q^* is the quantity that the firm chooses to maximize its profit. Explain why the following properties of the optimal profit function are true:

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1) optimal profit function: $\pi^*(w,p) = \max_q [pq - C(q)]$

For a price-taking firm (p is set by market) that uses only labor at wage w, with production $q = f(L)$ we can also write the optimal profit function in terms of labor:

$$\pi^*(w,p) = \max_{L \geq 0} [p \cdot f(L) - wL]$$

Let $L^*(w,p)$ be an optimizer and $q^*(w,p) = f(L^*)$

Homogeneous of degree 1: A function $g(x,y)$ is homogeneous of degree 1 when if you multiply all it's args by some positive constant, the output also gets multiplied by the constant

E.g. $g(tx, ty) = t\,g(x,y)$

If all prices in an economy double (both p and w double), each unit of output now sells for twice as much but each unit of labor also costs twice as much. Since everything is scaled up by same factor, the real situation hasn't changed — only the units of money has.

We found $\pi^*(w,p) = \frac{p^2}{4w}$

$$\Rightarrow \pi^*(tw, tp) = \frac{(tp)^2}{4tw} = \frac{t^2 p^2}{4tw} = t \cdot \frac{p^2}{4w} = t\,\pi^*(p,w)$$

2) p (market price) increases: $p_2 > p_1$

$\therefore$ for any L: $\pi^*(w,p_2) > \pi^*(w,p_1) \Rightarrow$ our optimal profit is now greater

3) $w_2 > w_1 \Rightarrow$ If w ↑ but everything remains the same: - Producing same q of goods is ↑expensive
- Profits will ↓

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(a) $f(l,k) = l^\alpha k^\beta$

b) $f(L,k) = \alpha L + \beta k$

c) $f(L,k) = \sqrt{L} + k$

a) $f(L,k) = L^{\alpha} k^{\beta}$

1) Solve for $k = g(L)$

$L^{\alpha} k^{\beta} = q_0$

$k^{\beta} = \frac{q_0}{L^{\alpha}}$

$k = \left(\frac{q_0}{L^{\alpha}}\right)^{\frac{1}{\beta}} = q_0^{\frac{1}{\beta}} \cdot L^{-\frac{\alpha}{\beta}}$

2) Derivative along isoquant

$\frac{dk}{dL} = q_0^{\frac{1}{\beta}} \left(-\frac{\alpha}{\beta}\right) L^{-\frac{\alpha}{\beta}-1}$

$k = q_0^{\frac{1}{\beta}} L^{-\frac{\alpha}{\beta}}$

$\frac{dk}{dL} = -\frac{\alpha}{\beta} \cdot \frac{k}{L}$

3) MRTS

$-\frac{dk}{dL} = \frac{\alpha}{\beta} \cdot \frac{k}{L} = \frac{MP_L}{MP_k}$

b) $f(L,k) = \alpha L + \beta k$

1) Isoquant ($k = g(L)$)

$\alpha L + \beta k = q_0$

$\beta k = q_0 - \alpha L$

$k = \frac{q_0 - \alpha L}{\beta}$

$k = \frac{q_0}{\beta} - \frac{\alpha L}{\beta}$

2) Derivative

$\frac{dk}{dL} = -\frac{\alpha}{\beta}$

3) MRTS

$MRTS = -\frac{dk}{dL} = \frac{\alpha}{\beta}$

c) $f(L,k) = \sqrt{L} + k$

1) Isoquant

$\sqrt{L} + k = q_0$

$k = q_0 - \sqrt{L}$

2) Derivative

$\frac{dk}{dL} = -\frac{1}{2} L^{-\frac{1}{2}}$

$= \frac{-1}{2\sqrt{L}}$

3) MRTS

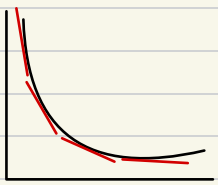
$MRTS = -\frac{dk}{dL} = \frac{1}{2\sqrt{L}}$

2. What are the relationship between the RTS and the first derivative of the function $g(l)$. Use the property to deduce that if the RTS is decreasing, then the function $g(l)$ (that represents the shape of the isoquant) is a convex function, i.e., its second derivative is positive.

We can rewrite $f(k,L)$ (an isoquant showing combinations of L and k that give output q_0) as $k = g(L)$ ⇒ For every level of labor L, the isoquant tells us how much k we need to keep output fixed

When we move along the isoquant, output doesn't change ⇒ $df = MP_L\, dL + MP_k\, dk = 0$. Rearranging this gives us $\frac{dk}{dL} = \frac{-MP_L}{MP_k}$. $\frac{dk}{dL}$ is the slope of the isoquant → $g'(L)$, so $g'(L) = \frac{-MP_L}{MP_k}$ and since we know $RTS = \frac{MP_L}{MP_k}$ ⇒ $RTS = -g'(L)$.

"RTS is ↓" means as we use more labor, each extra worker can replace less capital than before. Graphically this means the isoquant's slope becomes flatter

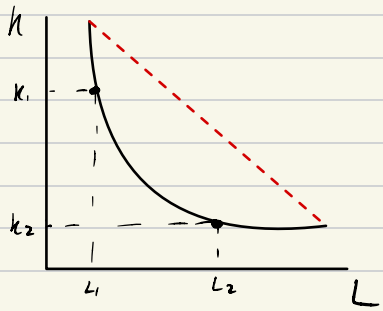


↓RTS (slope gets flatter)

3. If a function $g(x)$ is convex, then we have that for every $x_1 < x_2$ we have that for every $0 < \alpha < 1$ we have that

$$\alpha \cdot g(x_1) + (1-\alpha) \cdot g(x_2) > g\left(\alpha \cdot x_1 + (1-\alpha) \cdot x_2\right)$$

use the above result together with the result you found in clause 2 to show that if $f(l,k)$ exhibits diminishing RTS then for two bundles, A and B, along an isoquant, the bundle $C = rA + (1-r)B$, where $0 < r < 1$ lies on a higher isoquant.



$A = (L_1, k_1) \Rightarrow k_1 = g(L_1)$

$B = (L_2, k_2) \Rightarrow k_2 = g(L_2)$

$C = rA + (1-r)B$

$= r \cdot g(L_1) + (1-r)\, g(L_2)$

Let's use $f(L,K) = \sqrt{L} + k \Rightarrow MRTS = \frac{1}{2\sqrt{L}}$ ↓ in L

Fixed output, $q_0 = 10$

$\therefore \sqrt{L} + k = 10 \Rightarrow k = 10 - \sqrt{L}$ (convex: $g''(L) = \frac{1}{4}L^{-\frac{3}{2}} > 0$)

Pick 2 bundles: $-A = (9, 7)$ $(\sqrt{9} + 7 = 10)$

$-B = (16, 6)$ $(\sqrt{16} + 6 = 10)$

Midpoint mix $(r = 0.5)$: $C = (12.5, 6.5) \Rightarrow \left(\frac{16+9}{2}, \frac{6+7}{2}\right)$

$f(C) = \sqrt{12.5} + 6.5 = 10.035 > 10 \Rightarrow$ So C is on a higher isoquant

Convexity inequality: $0.5 \cdot g(9) + 0.5 \cdot g(16)$

$= 0.5 \cdot 7 + 0.5 \cdot 6 = 6.5$

$g(12.5) = 10 - \sqrt{12.5} = 6.46$

$6.5 > 6.46$, the mix is above the isoquant

(b) $f(l,k) = \alpha \cdot l + \beta \cdot k$
(c) $f(l,k) = \sqrt{l} + k$

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3. If a function $g(x)$ is convex, then we have that for every $x_1 < x_2$ we have that for every $0 < \alpha < 1$ we have that

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